EXERCISE – I HINTS & SOLUTIONS

Sol.1 A

$$\frac{1}{\sin\alpha}\int\frac{\sin\alpha}{\sin x.\sin(x+\alpha)}dx$$

$$\frac{1}{\sin\alpha}\int\frac{\sin\{(x+\alpha)-x\}}{\sin x.\sin(x+\alpha)}dx$$

$$\frac{1}{\sin\alpha}\int\frac{\sin(x+\alpha)\cos x-\cos(x+\alpha)\sin x}{\sin x\,\sin(x+\alpha)}dx$$

$$=\frac{1}{\sin\alpha}\int\cot x-\cot(x+\alpha)dx$$

$$=\frac{1}{\sin\alpha}[\ell n\sin x-\ell n\sin(x+\alpha)]+c$$

$$=\frac{1}{\sin\alpha}\left[\ell n\frac{\sin x}{\sin(x+\alpha)}\right]+c$$

Sol.2 B

$$\int\frac{a^{\sqrt{x}}}{\sqrt{x}}dx$$

put $\sqrt{x}=t$ $\Rightarrow \frac{1}{2\sqrt{x}}dx=dt \Rightarrow \frac{dx}{\sqrt{x}}=2dt$

$$=2\int a^t\,dt=\frac{2a^t}{\ell n\,a}+c=2\frac{a^{\sqrt{x}}}{\ell n\,a}+c$$

Sol.3 C

$$I=\int 5^{5^{5^x}}.5^{5^x}.5^x\,dx$$

Let $5^{5^{5^x}}=t$

$$5^{5^{5^x}}.\ell n5.5^{5^x}\,\ell n5.5^x\,\ell n5\,dx=dt$$

$$5^{5^{5^x}}.5^{5^x}.5^x\,dx=\frac{dt}{(\ell n5)^3}$$

$$I=\int\frac{dt}{(\ell n5)^3}=\frac{t}{(\ell n5)^3}+c=\frac{5^{5^{5^x}}}{(\ell n5)^3}+c$$

Sol.4 A

$$\int\frac{\sqrt{\tan x}}{\sin x\cos x}dx$$

$$\int\frac{\sqrt{\tan x}\sec^2 x}{\tan x}dx$$

$\tan x=t^2 \Rightarrow \sec^2 x\,dx=2t\,dt$

$$\int\frac{t.2t\,dt}{t^2}=2t+c=2\sqrt{\tan x}+c$$

Sol.5 D

$$\int\frac{2^x}{\sqrt{1-4^x}}dx$$

$2^x=t \Rightarrow 2^x\,\ell n2\,dx=dt \Rightarrow 2^x\,dx=\frac{dt}{\ell n2}$

$$\frac{1}{\ell n2}\int\frac{dt}{\sqrt{1-t^2}}=\frac{1}{\ell n2}\sin^{-1}(2^x)+c$$

Sol.6 D

$$y=\int\frac{dx}{(1+x^2)^{3/2}}$$

put $x=\tan\theta \Rightarrow dx=\sec^2\theta\,d\theta$

$$y=\int\frac{\sec^2\theta\,d\theta}{\sec^3\theta}=\int\cos\theta\,d\theta=\sin\theta+c$$

$y=\sin(\tan^{-1}x)+c$

$x=0,\ y=0 \Rightarrow c=0$

$y=\sin(\tan^{-1}x)$

$x=1 \Rightarrow y=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

Sol.7 B

$$y=\int\frac{dx}{x^2+x+1}$$

$$=\int\frac{dx}{x^2+x+\frac{1}{4}+1-\frac{1}{4}}=\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

Sol.8 C

$$\int (x-1)\, e^{-x}\, dx$$

$$= \int x\, e^{-x}\, dx - \int e^{-x}\, dx$$

$$= -xe^{-x} + \int e^{-x}\, dx - \int e^{-x}\, dx$$

$$= -xe^{-x} + c$$

Sol.9 C

$$\int \tan^3 2x \sec 2x\, dx$$

$$\int \tan 2x\, (\sec^2 2x - 1) \sec 2x\, dx$$

$$= \int \frac{\sin 2x}{\cos^4 2x}\, dx - \int \frac{\sin 2x}{\cos^2 2x}\, dx$$

put $\cos 2x = t$

$$\sin 2x\, dx = -\frac{dt}{2}$$

$$= -\frac{1}{2} \int \frac{dt}{t^4} + \frac{1}{2} \int \frac{dt}{t^2}$$

$$= -\frac{1}{2} \left[\frac{t^{-3}}{-3}\right] - \frac{1}{2}\, \frac{1}{t} + c$$

$$= \frac{1}{6} \sec^3 2x - \frac{1}{2} \sec 2x + c$$

Sol.10 A

$$\int e^{\tan^{-1}x} \left(\frac{1+x+x^2}{1+x^2}\right) dx$$

$$\tan^{-1} x = t \Rightarrow \frac{dx}{1+x^2} = dt$$

$$\int e^t (1 + \tan t + \tan^2 t)\, dt$$

$$= \int e^t (\sec^2 t + \tan t)\, dt$$

$$= e^t \tan t + c$$

$$= e^{\tan^{-1}x} \tan(\tan^{-1}x) + c = x\, e^{\tan^{-1}x} + c$$

Sol.11 D

$$\int \frac{dx}{x^2 (x^4+1)^{3/4}}$$

$$= \int \frac{dx}{x^5 (1+x^{-4})^{3/4}}$$

put $1 + x^{-4} = t^4 \Rightarrow \dfrac{dx}{x^5} = -t^3\, dt$

$$= -\int \frac{t^3\, dt}{t^3} = -t + c = -\left(1+\frac{1}{x^4}\right)^{1/4} + c$$

Sol.12 A

$$\int \frac{dx}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}$$

$$= \int \frac{1-\sin x}{\cos^2 x}\, dx$$

$$= \int \sec^2 x\, dx - \int \tan x \sec x\, dx$$

$$= \tan x - \sec x + c$$

$$= \frac{\sin x - 1}{\cos x} + c$$

$$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2} - \left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right)}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} + c$$

$$= -\frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}{\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)} + c$$

$$= -\frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)} + c = -\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right) + c$$

$$= -\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) + c = \tan\left(\frac{x}{2} - \frac{\pi}{4}\right) + c$$

$$a = -\frac{\pi}{4},\ b \in R$$

Sol.13 C

$$\int (f(x)\, g''(x) - f''(x)\, g(x))\, dx$$

$$= f(x) \int g''(x)\,.dx - \int f'(x)\, g'(x)\, dx$$

$$- g(x) \int f''(x)\, dx + \int g'(x)\, f'(x)\, dx$$

$$= f(x)\, g'(x) - f'(x)\, g(x) + c$$

Sol.14 B

$$\int (\sin 2x - \cos 2x)\, dx = \frac{1}{\sqrt{2}} \sin (2x - a) + b$$

$$= -\frac{\cos 2x}{2} - \frac{\sin 2x}{2} + b$$

$$= \frac{-1}{\sqrt{2}} \left[\frac{1}{\sqrt{2}} \cos 2x + \frac{1}{\sqrt{2}} \sin 2x\right] + b$$

$$= -\frac{1}{\sqrt{2}} \left[\sin \frac{\pi}{4} \cos 2x + \cos \frac{\pi}{4} \sin 2x\right] + b$$

$$= \frac{1}{\sqrt{2}} \left[\sin \left(2x + \frac{5\pi}{4}\right)\right] + b$$

Sol.15 B

$$\int \frac{\cos 2x}{(\cos x + \sin x)^2}\, dx$$

$$\int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx$$

$\cos x + \sin x = t \Rightarrow (-\sin x + \cos x)\, dx = dt$

$$\int \frac{dt}{t} = \ell n\, (\cos x + \sin x) + c$$

Sol.16 A

$$\int \frac{1}{x(x^n + 1)} dx = \int \frac{1}{x^{1+n}(1 + x^{-n})} dx$$

$$1 + x^{-n} = t \Rightarrow -nx^{-n-1}\, dx = dt \Rightarrow \frac{dx}{x^{n+1}} = \frac{-1}{n} dt$$

$$= -\frac{1}{n} \int \frac{dt}{t} \qquad = -\frac{1}{n}\, \ell n\, (1 + x^{-n}) + c$$

Sol.17 C

$$\int [1 + \tan x \tan(x + \alpha)]\, dx$$

$$\int \frac{\tan(x + \alpha) - \tan x}{\tan(x + \alpha - x)}\, dx$$

$$= \frac{1}{\tan\alpha} \int \tan\, (x + \alpha)\, dx - \int \tan x\, dx$$

$$= \frac{1}{\tan\alpha}\, \ell n \left|\frac{\sec\, (x + \alpha)}{\sec x}\right| + c$$

Sol.18 A

$$\int \sqrt{\frac{e^x - 1}{e^x + 1}}\, dx$$

$$= \int \frac{e^x - 1}{\sqrt{e^{2x} - 1}}\, dx$$

$$= \int \frac{e^x}{\sqrt{e^{2x} - 1}}\, dx - \int \frac{dx}{\sqrt{e^{2x} - 1}}$$

$$= \int \frac{dt}{\sqrt{t^2 - 1}} - \int \frac{e^x}{e^x \sqrt{e^{2x} - 1}}\, dx$$

$$= \int \frac{dt}{\sqrt{t^2 - 1}} - \int \frac{du}{u\sqrt{u^2 - 1}}$$

$$= \ell n \left(e^x + \sqrt{e^{2x} - 1}\right) - \sec^{-1} (e^x) + c$$

Sol.19 C

$$\int \frac{dx}{x^3(1 + x)} = \frac{A}{x^2} + \frac{B}{x} + \ell n \left(\frac{x}{x + 1}\right) + c$$

$$\frac{1}{x^3(1 + x)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} + \frac{d}{(x + 1)}$$

$1 = ax^2\, (x + 1) + bx\, (x + 1) + c(x + 1) + d\, x^3$

put $x = 0 \Rightarrow c = 1$

put $x = -1 \Rightarrow d = -1$

put $x = 1$

$1 = 2a + 2b + 2c + d$

$1 = 2a + 2b + 2 - 1 \Rightarrow a + b = 0$

put $x = 2$

$1 = 12a + 6b + 3c + 8d$

$1 = 12a + 6b + 3 - 8$

$12a + 6b = 6 \Rightarrow 2a + b = 1 \Rightarrow a = 1$

$$\int \frac{dx}{x^3(1+x)} = \int \left(\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x+1}\right) dx$$

$$= \ell n\, x + \frac{1}{x} - \frac{1}{2x^2} - \ell n\,(x + 1) + c$$

$$= -\frac{1}{2x^2} + \frac{1}{x} + \ell n \left(\frac{x}{x+1}\right) + c$$

A = – 1/2, B = 1

Aliter : $\int \frac{1+x^3-x^3}{x^3(1+x)} = \int \frac{1+x^3}{x^3(1+x)} - \int \frac{dx}{1+x}$

Sol.20 C

$$\int \sqrt{\sec x - 1}\, dx$$

$$\int \frac{\tan x \sec x}{\sqrt{\sec x + 1}\,.\,\sec x}\, dx$$

$\sec x + 1 = t^2 \Rightarrow \sec x \tan dx = 2t\, dt$

$$= \int \frac{2t\, dt}{t\,(t^2-1)} = 2\int \frac{dt}{t^2-1} = \ell n\, \frac{t-1}{t+1} + c$$

$$= \ell n\, \frac{\sqrt{\sec x + 1} - 1}{\sqrt{\sec x + 1} + 1} + c$$

Sol.21 B

$$\int \frac{dx}{\cos^3 x \sqrt{\sin 2x}}$$

$$\int \frac{dx}{\cos^3 x \sqrt{2 \sin x \cos x}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\cos^4 x \sqrt{\tan x}} = \frac{1}{\sqrt{2}} \int \frac{\sec^4 x}{\sqrt{\tan x}}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{(1+\tan^2 x)}{\sqrt{\tan x}} \sec^2 x\, dx$$

put $\tan x = t^2 \Rightarrow \sec^2 x\, dx = 2t\, dt$

$$= \frac{1}{\sqrt{2}} \int \left(\frac{1+t^4}{t}\right) 2t\, dt = \sqrt{2} \int (1+t^4)\, dt$$

$$= \sqrt{2} \left[t + \frac{t^5}{5}\right] + c$$

$$= \sqrt{2} \left[\tan^{1/2} x + \frac{\tan^{5/2} x}{5}\right] + c$$

Sol.22 C

$$\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}\, dx$$

Let $4e^x + 6e^{-x} = A(9e^x - 4e^{-x}) + B(9e^x + 4e^{-x})$

$4 = 9A + 9B$; $6 = -4A + 4B$

$B = \frac{35}{36}$; $A = -\frac{19}{36}$

Sol.23 B

$$f(x) = \int \frac{2\sin x - \sin 2x}{x^3}$$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{2\sin x - 2\sin x \cos x}{x^3}$$

$$= \lim_{x\to 0} \frac{2\sin x}{x} \left(\frac{1-\cos x}{x^2}\right)$$

$= 1$

Sol.24 B

$$\int \frac{\cos 4x + 1}{\cot x - \tan x} = A \cos 4x + B$$

$$I = \int \frac{(\cos 4x + 1)}{(\cos^2 x - \sin^2 x)} \cos x \sin x\, dx$$

$$= \int \left(\frac{2\cos^2 2x}{\cos 2x}\right) (\cos x \sin x)\, dx$$

$$= \int \cos 2x \sin 2x\, dx$$

$$= \frac{1}{2} \int \sin 4x\, dx = -\frac{\cos 4x}{8} + B$$

Sol.25 C

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} (x + \sqrt{x})\, dx$$

$$\sqrt{x} = t \Rightarrow \frac{1}{\sqrt{x}}\, dx = 2\, dt$$

$$2\int e^t\, (t^2 + t)\, dt$$

$$= 2\int e^t\, (t^2 + 2t) - 2\int e^t\, t\, dt$$

$= 2\, e^t\, (t^2) - 2[t\, e^t - e^t] + c$

$= 2e^{\sqrt{x}}.x - 2[\sqrt{x}\, e^{\sqrt{x}} - e^{\sqrt{x}}] + c$

$= 2e^{\sqrt{x}}[x - \sqrt{x} + 1] + c$

Sol.26 D

$\int e^{\tan\theta}(\sec\theta - \sin\theta)\, d\theta$

$\tan\theta = t \Rightarrow d\theta = \frac{dt}{1+t^2}$

$I = \int \frac{e^t}{1+t^2}\left(\sqrt{1+t^2} - \frac{t}{\sqrt{1+t^2}}\right) dt$

$= \int e^t \left(\frac{1}{\sqrt{1+t^2}} - \frac{t}{\sqrt{(t^2+1)^3}}\right) dt$

$= e^t \frac{1}{\sqrt{(t^2+1)}} + c \qquad = e^{\tan\theta} \frac{1}{\sec\theta} + c$

$= e^{\tan\theta} \cos\theta + c$

Sol.27 C

$\int \frac{1-x^7}{x(1+x^7)} dx$

$= \int \frac{1}{x(1+x^7)} dx - \int \frac{x^6}{(1+x^7)} dx$

$= \int \frac{1}{x^8(x^{-7}+1)} dx - \int \frac{x^6}{(1+x^7)} dx$

$1 + x^{-7} = t \qquad 1 + x^7 = u$

$\frac{-7}{x^8} dx = dt \qquad x^6\, dx = \frac{du}{7}$

$\frac{dx}{x^8} = \frac{-1}{7} dt$

$= -\frac{1}{7}\int \frac{dt}{t} - \frac{1}{7}\int \frac{du}{u} = -\frac{1}{7}\ell n\, t - \frac{1}{7}\ell n\, u$

$= -\frac{1}{7}\ell n\,(1 + x^{-7}) - \frac{1}{7}\ell n\,(1 + x^7) + c$

Sol.28 D

$\int \sqrt{\frac{1-\cos x}{\cos\alpha - \cos x}}\, dx$

$= \int \sqrt{\frac{1-\left(1-2\sin^2\frac{x}{2}\right)}{\cos\alpha - \left(2\cos^2\frac{x}{2} - 1\right)}}\, dx$

$= \sqrt{2}\int \frac{\sin\frac{x}{2}}{\sqrt{\cos\alpha + 1 - 2\cos^2\frac{x}{2}}}\, dx$

$= \sqrt{2}\int \frac{\sin\frac{x}{2}}{\sqrt{2\cos^2\frac{\alpha}{2} - 2\cos^2\frac{x}{2}}}\, dx$

$= \int \frac{\sin\frac{x}{2}}{\sqrt{\cos^2\frac{\alpha}{2} - \cos^2\frac{x}{2}}}\, dx$

$\cos\frac{x}{2} = t \Rightarrow -\frac{1}{2}\sin\frac{x}{2}\, dx = dt$

$\sin\frac{x}{2}\, dx = -2dt$

$= -2\int \frac{dt}{\sqrt{\cos^2\frac{\alpha}{2} - t^2}} = -2\sin^{-1}\frac{t}{\cos\frac{\alpha}{2}} + c$

$= -2\sin^{-1}\frac{\cos\frac{x}{2}}{\cos\frac{\alpha}{2}} + c$

Sol.29 A

$\int \frac{dx}{[(x-1)^3 (x+2)^5]^{1/4}}$

$= \int \frac{dx}{(x-1)^{3/4}(x+2)^{5/4}}$

$x - 1 = t^4 \Rightarrow dx = 4t^3\, dt$

$\int \frac{4t^3\, dt}{t^3 (t^4+3)^{5/4}} = 4\int \frac{dt}{(t^4+3)^{5/4}}$

$= 4\int \frac{dt}{t^5(1+3t^{-4})^{5/4}}$

$1 + 3t^{-4} = z^4 \Rightarrow -\frac{4}{t^5}\,dt = \frac{4}{3}z^3\,dz$

$= -\frac{4}{3}\int \frac{z^3\,dz}{z^5} = \frac{4}{3}\,\frac{1}{z} = \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + c$

Sol.30 C

$\int (x\,e^{\ell n \sin x} - \cos x)\,dx$

$= \int (x \sin x - \cos x)\,dx$

$= -x \cos x + c$

$= -e^{\ell n\, x} \cos x + c$

Sol.31 A

$\int \frac{\sin^2 x}{1+\sin^2 x}\,dx = \int \frac{\sin^2 x + 1 - 1}{1+\sin^2 x}\,dx$

$= \int dx - \int \frac{dx}{1+\sin^2 x}$

$= \int dx - \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x}\,dx$

$= \int dx - \int \frac{\sec^2 x}{1+2\tan^2 x}\,dx$

$= \int dx - \int \frac{dt}{1+2t^2}$

$= x - \left(\frac{1}{\sqrt{2}}\right)\tan^{-1}\sqrt{2}\,t + c$

$= x - \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\,\tan x) + c$

Sol.32 B

$\int 4 \sin x \cos\frac{x}{2}\cos\frac{3x}{2}\,dx$

$2\int\left(2 \sin x \cos\frac{x}{2}\right)\cos\frac{3x}{2}\,dx$

$2\int\left(\sin\frac{3x}{2} + \sin\frac{x}{2}\right)\cos\frac{3x}{2}\,dx$

$= \int 2\sin\frac{3x}{2}\cos\frac{3x}{2}\,dx + \int 2\sin\frac{x}{2}\cos\frac{3x}{2}\,dx$

$= \int \sin 3x\,dx + \int ((\sin 2x) - \sin x)\,dx$

$= -\frac{\cos 3x}{2} - \frac{\cos 2x}{2} + \cos x + c$

Sol.33 A

$\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

$= \int \frac{1-\sqrt{x}}{\sqrt{1-x}}\,dx$

$= \int \frac{dx}{\sqrt{1-x}} - \int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx$

Let $I_1 = \int \sqrt{\frac{x}{1-x}}\,dx = \int \frac{\sqrt{x}}{\sqrt{1-(\sqrt{x})^2}}\,dx$

put $\sqrt{x} = t \Rightarrow \frac{dx}{2\sqrt{x}} = dt$

$= \int \frac{2t^2\,dt}{\sqrt{1-t^2}} = 2\int \frac{t^2+1-1}{\sqrt{1-t^2}}\,dt$

$= 2\int \frac{dt}{\sqrt{1-t^2}} - 2\int \sqrt{1-t^2}\,dt$

$= 2\sin^{-1} t - 2\left[\frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t\right] + c$

$I = -2\sqrt{1-x} - 2\sin^{-1}\sqrt{x} + \sqrt{x}\,\sqrt{1-x} + \sin^{-1}\sqrt{x} + c$

$= -2\sqrt{1-x} - \sin^{-1}\sqrt{x} + \sqrt{x}\,\sqrt{1-x} + c$

$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}\,\sqrt{1-x} + c$

Sol.34 B

$\int \sin x.\cos x.\cos 2x \cos 4x.\cos 8x.\cos 16x\,dx$

$= \frac{1}{2}\int (\sin 2x.\cos 2x)\cos 4x.\cos 8x.\cos 16x\,dx$

$= \frac{1}{4}\int (\sin 4x.\cos 4x)\cos 8x.\cos 16x\,dx$

$= \frac{1}{8}\int (\sin 8x.\cos 8x)\cos 16x\,dx$

$$= \frac{1}{16}\int \sin 16x \cos 16x \, dx$$

$$= \frac{1}{32}\int \sin 32x \, dx = -\frac{1}{32} \times \frac{\cos 32x}{32}$$

$$= -\frac{1}{1024}\cos 32x + c$$

Sol.35 C

$$\int \frac{dx}{\sin^6 x + \cos^6 x}$$

$$= \int \frac{dx}{\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x}$$

$$= \int \frac{\sec^4 x \, dx}{\tan^4 x + 1 - \tan^2 x}$$

$$= \int \frac{(1+\tan^2 x)\sec^2 x \, dx}{\tan^4 x - \tan^2 x + 1}$$

$\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$$= \int \frac{(1+t^2)\,dt}{t^4 - t^2 + 1} = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2 + 1} dt$$

Let $t - \frac{1}{t} = u \Rightarrow \left(1+\frac{1}{t^2}\right) dt = du$

$$= \int \frac{du}{1+u^2} = \tan^{-1} u + c = \tan^{-1}\left(t - \frac{1}{t}\right) + c$$

$$= \tan^{-1}\left(\tan x - \frac{1}{\tan x}\right) + c$$

$= \tan^{-1}(\tan x - \cot x) + c$

Sol.36 A

$$\int \left\{ \underbrace{\ell n(1+\sin x)}_{f(x)} + \underbrace{x \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)}_{f'(x)} \right\} dx$$

$= x\,\ell n\,(1 + \sin x) + c$

Sol.37 C

$$\int \sqrt{\frac{x-1}{x+1}} \cdot \frac{1}{x^2} dx = \int \sqrt{\frac{1-1/x}{1+1/x}} \cdot \frac{1}{x^2} dx$$

$$\frac{1}{x} = t \Rightarrow \frac{dx}{x^2} = -dt$$

$$\int -\sqrt{\frac{1-t}{1+t}} \, dt$$

put $t = \cos 2\theta \Rightarrow dt = -2 \sin 2\theta \, d\theta$

$$= \int 2\frac{\sin\theta}{\cos\theta}(2\sin\theta\cos\theta)\,d\theta$$

$$= 2\int 2\sin^2\theta \, d\theta = 2\int (1-\cos 2\theta)\,d\theta$$

$$= 2\theta - \sin 2\theta + c = \cos^{-1} t - \sqrt{1-t^2} + c$$

$$= \cos^{-1}\frac{1}{x} - \sqrt{1-\frac{1}{x^2}} + c = \sec^{-1} x - \frac{\sqrt{x^2-1}}{x} + c$$

Sol.38 B

$$\int \frac{dx}{\cos^3 x \sqrt{\sin 2x}}$$

same as question no.21

Sol.39 A

$$\int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}} = a\sqrt{\cot x} + b\sqrt{\tan^3 x} + c$$

$$= \int \frac{\sec^4 x \, dx}{\sqrt{\tan^3 x}}$$

$\tan x = t^2 \Rightarrow \sec^2 x \, dx = 2t \, dt$

$$\int \frac{(1+\tan^2 x)}{\tan^{3/2} x} \sec^2 x \, dx$$

$$= \int \left(\frac{1+t^4}{t^3}\right) 2t \, dt = 2\int \left(\frac{1}{t^2} + t^2\right) dt$$

$$= -\frac{2}{t} + \frac{2}{3}t^3 + c = -2\sqrt{\cot x} + \frac{2}{3}\sqrt{\tan^3 x} + c$$

Sol.40 A

$$\int \left\{\frac{(\log x - 1)}{1+(\log x)^2}\right\}^2 dx$$

$\ell n\, x = t \Rightarrow x = e^t \Rightarrow dx = e^t \, dt$

$$= \int e^t \left(\frac{t-1}{t^2+1}\right)^2 dt = \int e^t \left(\frac{t^2+1-2t}{(t^2+1)^2}\right) dt$$

$$= \int e^t \left\{ \underset{\underset{f(t)}{\uparrow}}{\frac{1}{(t^2+1)}} - \underset{\underset{f'(t)}{\uparrow}}{\frac{2t}{(1+t^2)^2}} \right\} dt$$

$$= \frac{e^t}{1+t^2} = \frac{x}{1+\log^2 x} + c$$

Sol.41 B

$$\int \frac{\sin x}{\sin(x-a)} dx$$

Let $x - a = t \Rightarrow dx = dt$

$$\int \frac{\sin(a+t)}{\sin t} dt$$

$$= \int \frac{\sin a \cos t + \cos a \sin t}{\sin t} dt$$

$$= \sin a \int \cot t\, dt + \cos a \int dt$$

$= t \,.\, \cos a + \sin a \,.\, \ell n (\sin t) + c$
$= (x - a) \cos a + \sin a\ \ell n \sin (x - a) + c$
$= x \cos a + \sin a\ \ell n \sin (x - a) + c$
$(\cos \alpha, \sin \alpha)$

Sol.42 A

$$\int \frac{dx}{\cos x - \sin x} = \int \frac{\left(1+\tan^2 \frac{x}{2}\right)}{1-\tan^2 \frac{x}{2} - 2\tan\frac{x}{2}}$$

$$= \int \frac{\sec^2 \frac{x}{2} dx}{1-\tan^2 \frac{x}{2} - 2\tan\frac{x}{2}}$$

Put $\tan \frac{x}{2} = t \Rightarrow \sec^2 \frac{x}{2} dx = 2dt$

$$= 2\int \frac{dt}{1-t^2-2t} = 2\int \frac{-dt}{(t^2+2t-1)}$$

$$= 2\int \frac{-dt}{(t^2+2t+1-2)} = 2\int \frac{dt}{2-(t+1)^2}$$

$$= \frac{2}{2\sqrt{2}}\ \ell n\ \frac{\sqrt{2}+t+1}{\sqrt{2}-(t+1)} + c$$

$$= \frac{1}{\sqrt{2}}\ \ell n\ \frac{\sqrt{2}+\tan\frac{x}{2}+1}{\sqrt{2}-\left(\tan\frac{x}{2}+1\right)} + c$$

Sol.43 A

$$\int \frac{dx}{x+x^5} = f(x) + c$$

$$= \int \frac{x^4+1-1}{x+x^5} dx = \int \frac{dx}{x} - \int \frac{dx}{x+x^5}$$

$= \ell n\ x - f(x) + c$

Sol.44 B

$$\int \frac{3x^4-1}{(x^4+x+1)^2} dx = \int \frac{3x^4-1}{x^2\left(x^3+1+\frac{1}{x}\right)^2} dx$$

$$= \int \frac{3x^2-1/x^2}{\left(x^3+1+\frac{1}{x}\right)^2} dx$$

$x^3 + 1 + \frac{1}{x} = t \Rightarrow \left(3x^2 - \frac{1}{x^2}\right) dx = dt$

$$= \int \frac{dt}{t^2} = -\frac{1}{t} + c$$

$$= -\frac{1}{\left(x^3+1+1/x\right)} + c = -\frac{x}{x^4+x+1} + c$$

Sol.45 C

$$\int \frac{x^4+1}{x(x^2+1)^2} dx$$

$$\Rightarrow \int \left(\frac{x^4+2x^2+1-2x^2}{x(x^2+1)^2}\right) dx$$

$$\Rightarrow \int \frac{dx}{x} + \int \frac{dt}{t^2} \Rightarrow \ell n\ x + \frac{1}{1+x^2} + c$$

Sol.46 A

$$I = \int \underbrace{\left(\frac{x}{\sqrt{1+x^2}}\right)}_{\text{II}} \underbrace{\ell n\left(x+\sqrt{1+x^2}\right)}_{\text{I}} dx$$

$$= \ell n(x + \sqrt{1+x^2}) \int \frac{x}{\sqrt{1+x^2}}dx - \int \frac{1}{\left(x+\sqrt{1+x^2}\right)}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$I_1 = \int \frac{x}{\sqrt{1+x^2}}dx$$

$1 + x^2 = t^2 \Rightarrow xdx = tdt$

$$I = \ell n\,(x+\sqrt{1+x^2}) \int \frac{t\,dt}{t} - \int \left(\frac{1}{\sqrt{1+x^2}} \int dt\right) dx$$

$$= \ell n\,(x + \sqrt{1+x^2}) \,.\, \sqrt{1+x^2} - x + c$$

Sol.47 A

$$\int \frac{1}{x\sqrt{1-x^3}}\,dx$$

$1 - x^3 = t^2$ (Aliter put $x^3 = \sin^2\theta$)

$$x^2\,dx = -\frac{2}{3}tdt$$

$$= \int \frac{x^2\,dx}{x^3\sqrt{1-x^3}}$$

$$= -\frac{2}{3}\int \frac{t\,dt}{(1-t^2)\,t} = \frac{2}{3}\int \frac{dt}{(t^2-1)}$$

$$= \frac{2}{3}\,\ell n\,\frac{t-1}{t+1} + c = \frac{1}{3}\,\ell n\,\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1} + c$$

Sol.48 B

$$\int \frac{\cos^3 x}{\sin^2 x + \sin x}\,dx = \int \frac{(1-\sin^2 x)\cos x}{\sin x(1+\sin x)}dx$$

put $\sin x = t \Rightarrow \cos x\,dx = dt$

$$\int \frac{(1-t^2)}{t(1+t)}\,dt = \int \frac{(1-t)}{t}\,dt$$

$= \ell n\, t - t + c = \ell n\,|\sin x| - \sin x + c$

Aliter $$\int \frac{\cos^3 x}{\sin^2 x + \sin x}\,dx$$

$$= \int \frac{\cos^3 x}{\sin^4 x - \sin^2 x}\,(\sin^2 x - \sin x)\,dx$$

$$= \int \cos x\left(\frac{1}{\sin x} - 1\right)dx$$

put $\sin x = t \Rightarrow \cos x\,dx = dt$

$$= \int \left(\frac{1}{t}-1\right)dt = \ell n\,|\,t\,| - t = \ell n\,|\sin x| - \sin x + c$$

Sol.49 A

$$\int \frac{dx}{\sqrt{\sin^3 x \cos x}}$$

$$= \int \frac{dx}{\sqrt{\tan^3 x . \cos^4 x}}$$

$$= \int \frac{\sec^2 x\,dx}{\sqrt{\tan^3 x}}$$

$\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$= \int \frac{dt}{t^{3/2}} = \frac{t^{-3/2+1}}{1-3/2} + c = \frac{-2}{\sqrt{\tan x}} + c$$

Sol.50 B

$$\int \frac{x^3-1}{x(x^2+1)}\,dx = \int \frac{x^2}{x^2+1}dx - \int \frac{1}{x(x^2+1)}dx$$

$$= \int dx - \int \frac{dx}{x^2+1} - \int \frac{1}{x^3(1+x^{-2})}\,dx$$

Let $1 + x^{-2} = t \Rightarrow \frac{dx}{x^3} = -\frac{dt}{2}$

$$= x - \tan^{-1} x + \frac{1}{2}\int \frac{dt}{t}$$

$$= x - \tan^{-1} x + \frac{1}{2}\,\ell n\,|1 + x^{-2}| + c$$

$$= x - \tan^{-1} x + \frac{1}{2}\,\ell n\,(x^2 + 1) - \ell n\,x + c$$

Sol.51 A

$$\int \frac{\ell n|x|}{x\sqrt{1+\ell n\, x}}\,dx$$

$1 + \ell n\, x = t^2 \Rightarrow \frac{1}{x}dx = 2t\, dt$

$$\int \frac{(t^2-1)\,2t\,dt}{t} = 2\int (t^2-1)\,dt$$

$$= 2\left[\frac{t^3}{3} - t\right] + c = \frac{2}{3}t\,[t^2-3] + c$$

$$= \frac{2}{3}\sqrt{1+\ell n\, x}\;[1 + \ell n\, x - 3] + c$$

$$= \frac{2}{3}\sqrt{1+\ell n\, x}\;[\ell n\, x - 2] + c$$

Sol.52 A

$$\int \frac{x\tan^{-1}x}{\sqrt{1+x^2}}\,dx = \sqrt{1+x^2}\,f(x) + A\,\ell n\,(x + \sqrt{1+x^2}) + c$$

$$\tan^{-1}x\int\frac{x}{\sqrt{1+x^2}}dx - \int\left(\frac{1}{1+x^2}\int\frac{x}{\sqrt{1+x^2}}dx\right)dx$$

Let $1 + x^2 = t^2 \Rightarrow xdx = tdt$

$$\tan^{-1}x\int\frac{t\,dt}{t} - \int\frac{1}{1+x^2}\left(\int dt\right)dx$$

$$= \sqrt{1+x^2}\;\tan^{-1}x - \int\frac{\sqrt{1+x^2}}{1+x^2}dx$$

$$= \tan^{-1}x\,\sqrt{1+x^2} - \int\frac{dx}{\sqrt{1+x^2}}$$

$$= \tan^{-1}x\,\sqrt{1+x^2} - \ell n\,(x + \sqrt{x^2+1}) + c$$

$f(x) = \tan^{-1}x$

$A = -1$

Sol.53 B

$$\int\frac{\sin^8x - \cos^8x}{1 - 2\sin^2x\cos^2x}\,dx$$

$$\int\frac{(\sin^4 + \cos^4x)\,(\sin^2x + \cos^2x)\,(\sin^2x - \cos^2x)}{(\sin^4x + \cos^4x)}\,dx$$

$$= -\int\cos 2x\,dx = -\frac{\sin 2x}{2} + c$$

Sol.54 C

$$\int\{1 + 2\tan x\,(\tan x + \sec x)\}^{1/2}\,dx$$

$$\int\{1 + 2\tan^2x + 2\tan x\sec x\}^{1/2}\,dx$$

$$\int\{\sec^2x - \tan^2x + 2\tan^2x + 2\tan x\sec x\}^{1/2}\,dx$$

$$= \int(\sec x + \tan x)\,dx$$

$= \ell n\sec x + \ell n\,(\sec x + \tan x) + c$

$= \ell n\sec x\,(\sec x + \tan x) + c$

Sol.55 B

$$\int\frac{x\,dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$$

Let $1 + x^2 = t^2 \Rightarrow xdx = tdt$

$$\int\frac{t\,dt}{\sqrt{t^2+t^3}} = \int\frac{dt}{\sqrt{1+t}}$$

$$= 2\sqrt{1+t} + c = 2\sqrt{1+\sqrt{1+x^2}} + c$$

Sol.56 B

$$\int\frac{1+x^4}{(1-x^4)^{3/2}}\,dx$$

$$= \int\frac{1+x^4}{x^3\left(\frac{1}{x^2}-x^2\right)^{3/2}}\,dx = \int\frac{\frac{1}{x^3}+x}{\left(\frac{1}{x^2}-x^2\right)^{3/2}}\,dx$$

Let $\frac{1}{x^2} - x^2 = t^2 \Rightarrow \left(\frac{1}{x^3}+x\right)dx = -t\,dt$

$$= -\int\frac{t\,dt}{t^3} = -\left(\frac{t^{-2+1}}{-2+1}\right) + c$$

$$= \frac{1}{t} + c = \frac{1}{\sqrt{\frac{1}{x^2}-x^2}} + c$$

Sol.57 A

$$\int \left(\sqrt{\frac{a+x}{a-x}} - \sqrt{\frac{a-x}{a+x}}\right) dx = \int \frac{(a+x)-(a-x)}{\sqrt{a^2-x^2}}\, dx$$

$$= \int \frac{2x}{\sqrt{a^2-x^2}}\, dx$$

$a^2 - x^2 = t^2 \Rightarrow xdx = tdt$

$$= -2\int \frac{t\,dt}{t} = -2t + c = -2\sqrt{a^2-x^2} + c$$

Sol.58 B

$$\int \tan(x-\alpha)\tan(x+\alpha)\tan 2x\, dx$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}$$

$$\tan A \tan B = 1 - \frac{(\tan A + \tan B)}{\tan(A+B)}$$

$$\tan(x-\alpha)\tan(x+\alpha) = 1 - \frac{(\tan(x+\alpha)+\tan(x-\alpha))}{\tan 2x}$$

$\tan 2x \tan(x-\alpha)\tan(x+\alpha)$
$= \tan 2x - \tan(x+\alpha) - \tan(x-\alpha)$

$$\int \tan 2x \tan(x-\alpha)\tan(x+\alpha)\, dx$$

$$= \int \tan 2x\, dx - \int \tan(x+\alpha)\, dx - \tan(x-\alpha)dx$$

$$= \frac{1}{2}\ell n \sec 2x - \ell n \sec(x+a) - \ell n \sec(x-\alpha) + c$$

$$= \ell n \frac{\sqrt{\sec 2x}}{\sec(x+\alpha)\sec(x-\alpha)} + c$$

Sol.59 C

$$\int x^{13/2}(1+x^{5/2})^{1/2}\, dx$$

$$1 + x^{5/2} = t^2 \Rightarrow x^{3/2}\, dx = \frac{4}{5}\, t\, dt$$

$$\int x^5 . x^{3/2}(1+x^{5/2})^{1/2}\, dx$$

$$= \frac{4}{5}\int (t^2-1)^2 . t^2\, dt = \frac{4}{5}\int (t^4-2t^2+1)t^2\, dt$$

$$= \frac{4}{5}\int (t^6-2t^4+t^2)\, dt = \frac{4}{5}\left[\frac{t^7}{7} - \frac{2}{5}t^5 + \frac{t^3}{3}\right] + c$$

$$= \frac{4}{35}(1+x^{5/2})^{7/2} - \frac{8}{25}(1+x^{5/2})^{5/2} + \frac{4}{15}(1+x^{5/2})^{3/2} + c$$

Sol.60 A

$$2\int \frac{\sin x}{\sin 4x}\, dx$$

$$2\int \frac{\sin x}{2\sin 2x \cos 2x} dx = \int \frac{\sin x}{2\sin x \cos x \cos 2x}\, dx$$

$$= \frac{1}{2}\int \frac{dx}{\cos x \cos 2x} = \frac{1}{2}\int \frac{\cos x\, dx}{\cos 2x \cos^2 x}$$

$$= \frac{1}{2}\int \frac{\cos x\, dx}{(1-2\sin^2 x)(1-\sin^2 x)}$$

put $\sin x = t \Rightarrow \cos x\, dx = dt$

$$= \frac{1}{2}\int \frac{dt}{(1-2t^2)(1-t^2)} = \frac{1}{2}\int \frac{dt}{(t^2-1)(2t^2-1)}$$

$$= \frac{1}{2}\int \frac{(2t^2-1)-2(t^2-1)}{(t^2-1)(2t^2-1)}\, dt$$

$$= \frac{1}{2}\int \frac{dt}{t^2-1} - \frac{1}{2}\int \frac{dt}{t^2-1/2}$$

$$= \frac{1}{4}\,\ell n \frac{t-1}{t+1} - \frac{1}{2.2\left(1/\sqrt{2}\right)}\,\ell n \frac{t-1/\sqrt{2}}{t+1/\sqrt{2}} + c$$

$$= \frac{1}{4}\,\ell n \frac{\sin x - 1}{\sin x + 1} - \frac{1}{2\sqrt{2}}\,\ell n \frac{\sqrt{2}\sin x - 1}{\sqrt{2}\sin x + 1} + c$$

Sol.61 D

$$\int \frac{\tan^{-1}x - \cot^{-1}x}{\tan^{-1}x + \cot^{-1}x}\, dx$$

$$\Rightarrow \int \frac{\tan^{-1}x - \cot^{-1}x}{\pi/2}\, dx$$

$$\Rightarrow \frac{2}{\pi}\int \left(\tan^{-1}x - \frac{\pi}{2} + \tan^{-1}x\right) dx$$

$$\Rightarrow \frac{2}{\pi}\int \left(2\tan^{-1}x - \frac{\pi}{2}\right) dx$$

$$\Rightarrow \frac{4}{\pi}\left[x\tan^{-1}x - \frac{1}{2}\ell n|1+x^2|\right] - x + c$$

$$\Rightarrow \frac{4}{\pi}\, x\tan^{-1}x - \frac{2}{\pi}\,\ell n|1+x^2| - x + c$$